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COMMUNICATION

MORE ODD GRAPH THEORY

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Let k and n be positive integers with $n \geq 2k+1$, $k \geq 2$. We denote by $K = K(n, k)$ the graph with the k element subsets of $\{1, \ldots, n\}$ as vertices, where two such vertices are adjacent if they are disjoint. We determine the values of k and n for which K is a Cayley graph. In particular K is not a Cayley graph when $n = 2k+1$. This answers N.L. Biggs' question as to whether any of the "odd graphs" are Cayley graphs.

Introduction

Let k and n be positive integers with $n \geq 2k+1$, $k \geq 2$. We define $K = K(n, k)$ to be the graph with the k element subsets of $\{1, 2, \ldots, n\}$ as vertices, where two such vertices are adjacent if they are disjoint as sets. These graphs are commonly referred to as *Kneser's graphs*. If $n = 2k+1$ then K is also referred to as the *odd graph* O_{k+1}.

In [3], N. Biggs asks whether there are any values of k for which the odd graphs are Cayley graphs (for a definition of the term "Cayley graph", see chapter 16 of [2]). The following result provides the answer.

Theorem. *Except in the following cases, $K(n, k)$ is not a Cayley graph.*

(1) $k = 2$, *n is a prime-power and* $n \equiv 3 \pmod 4$.

(2) $k = 3$, $n = 8$ *or* 32.

Furthermore, if $n = 2k+1$ and $k \neq 2$ or 4, a transitive subgroup of $\Gamma = \mathrm{Aut}(K)$ is isomorphic (as an abstract group) to S_n or A_n.

Proof. From the solution to exercise 15.2 in [6] we know that if $\phi \in \Gamma$ and $x, y \in V(K)$, then

$$|x \cap y| = |x\phi \cap y\phi|.$$

Using this it is easy to show that Γ is just S_n, acting on $V(K)$ in the representation induced by its action on $\{1, 2, \ldots, n\}$.

A graph X is a Cayley graph for some group if and only if $\mathrm{Aut}(X)$ contains a subgroup acting regularly on $V(X)$. (See Lemma 4 of [7].) We will show that, except in cases (1) and (2), Γ contains no regular subgroup.

Let G be a transitive subgroup of Γ. The isomorphism from Γ to S_n takes G onto a subgroup $\bar{G}$ of S_n which acts transitively on the k-subsets of $\{1, 2, \ldots, n\}$. Now if $\bar{G}$ is actually k-transitive, then $n!/(n-k)!$ divides $|G| = |\bar{G}|$. Since $|V(K)| = \binom{n}{k}$, it follows that the vertex-stabilizer of G has order at least $k!$. Therefore if $\bar{G}$ is k-transitive, G is not regular.

A group which acts transitively on the k element subsets of a set is said to be *k-homogeneous*. By Theorem 2 of [5], when $k \geq 5$ a k-homogeneous group is k-transitive. Consequently if $k \geq 5$, Γ contains no regular subgroup. Suppose $k \leq 5$. In this case the k-homogeneous groups which are not k-transitive have been determined by W. Kantor in [4]. Comparing the orders of these groups with $\binom{n}{k}$ yields the conclusion that unless (1) or (2) hold, Γ contains no regular subgroup.

Assume now that $n = 2k+1$. Let G be a transitive subgroup of Γ and let $\bar{G}$ be defined as above. Then $\bar{G}$ is k-homogeneous and so, by Theorem 2 of [5], it is j-homogeneous for all such that $1 \leq j \leq k$. Further if $\bar{G}$ is j-homogeneous then it is $(n-j)$-homogeneous. Thus $\bar{G}$ is j-homogeneous for all j, $1 \leq j \leq n-1$. By the results of Beaumont and Peterson [1], we conclude that either G contains the alternating group A_n (and so $|\Gamma : G| \leq 2$) or $k = 2$ or 4.

The case $k = 2$ of the above theorem was first settled by Sabidussi [8]. (Note that when $k = 2$, K is isomorphic to the complement of the line graph of K_n.) Sabidussi defined the *deficiency* $d(X)$ of a graph X to be the least order of a vertex-stabilizer of a vertex transitive subgroup of Aut(X). He showed that graphs with arbitrarily large deficiency existed, but in the examples he used, $d(X)/|X|$ tended to zero as $|X|$ increased.

Our theorem shows that $d(O_k)/|O_k|$ increases with k, and thus provides a strengthening of Sabidussi's result.

We finish by drawing the reader's attention to the infinite family of 4-transitive cubic graphs referred to in chapter 18 of [2]. These graphs have $p(p^2-1)/48$ vertices, where p is a prime and $p \equiv \pm 1 \pmod{16}$. Let X_p be the graph corresponding to the prime p. Then it can be shown that $\mathrm{Aut}(X_p) \cong \mathrm{PSL}(2, p)$ and the vertex stabilizers in $\mathrm{Aut}(X_p)$ are isomorphic to S_4. Since $\mathrm{PSL}(2, p)$ has no subgroups with index less than $p+1$, it follows that the deficiency of these graphs is 24 when $p > 24$.

References

[1] R.A. Beaumont and R.P. Peterson, Set-transitive permutation groups, Canadian J. Math. 7 (1955) 35–42.

[2] N.L. Biggs, Algebraic Graph Theory, Cambridge Tracts in Mathematics, No. 67 (Cambridge University Press, London, 1974).

[3] N.L. Biggs, Some odd graph theory, In: Second International Conference on Combinatorial Mathematics, Annals of the New York Academy of Sciences Vol. 319 (New York Academy of Sciences, New York, 1979), 71–85.

[4] W. Kantor, k-Homogeneous groups, Math. Z. 124 (1972) 261–265.

[5] D. Livingston and A. Wagner, Transitivity of finite permutation groups, Math. Z. 90 (1965) 393–403.

[6] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).

[7] G. Sabidussi, On a class of fixed-point free graphs, Proc. Amer. Math. Soc. 9 (1958) 800–804.

[8] G. Sabidussi, Vertex-transitive graphs, Monatshefte für Math. 68 (1964) 426–438.